

### C3 PROOF

### Answers - Worksheet B

- 1 a e.g.  $x = \frac{1}{8} \Rightarrow \sqrt[3]{x} = \frac{1}{2}, \frac{1}{2} > \frac{1}{8}$   
 [ any value of  $x$  in the interval  $0 < x < 1$  ]
- b e.g.  $n = 7 \Rightarrow n^3 - n + 7 = 7(49 - 1 + 1)$  which is divisible by 7  
 [ many other examples ]
- 2 assume  $\sqrt{\pi}$  is rational  $\Rightarrow \sqrt{\pi} = \frac{p}{q}, p, q \in \mathbb{Z}$   
 $\Rightarrow \pi = \frac{p^2}{q^2}, p^2, q^2 \in \mathbb{Z} \therefore \pi$  rational  
 $\Rightarrow$  contradiction  $\therefore \sqrt{\pi}$  irrational
- 3 consider  $15x^2 - 11x + 2 < 0$   
 $\Rightarrow (5x - 2)(3x - 1) < 0$   
 $\Rightarrow \frac{1}{3} < x < \frac{2}{5}$
- e.g.  $x = 0.35 \Rightarrow 15x^2 - 11x + 2 = -0.0125, -0.0125 < 0$   
 [ any value of  $x$  in the interval  $\frac{1}{3} < x < \frac{2}{5}$  ]
- 4 a  $n^2 + 2n = (2m + 1)^2 + 2(2m + 1)$   
 $= 4m^2 + 4m + 1 + 4m + 2$   
 $= 4m^2 + 8m + 3$
- b assume  $n^2 + 2n$  even and  $n$  odd, where  $n \in \mathbb{Z}$   
 $n$  odd  $\Rightarrow n = 2m + 1, m \in \mathbb{Z}$   
 $\Rightarrow n^2 + 2n = 4m^2 + 8m + 3 = 2(2m^2 + 4m + 1) + 1$   
 $2m^2 + 4m + 1 \in \mathbb{Z} \therefore n^2 + 2n$  odd  
 $\Rightarrow$  contradiction  $\therefore n$  even
- 5 a  $k \cos x - \operatorname{cosec} x = 0 \Rightarrow k \cos x = \frac{1}{\sin x}$   
 $\Rightarrow k \sin x \cos x = 1$   
 $\Rightarrow \frac{1}{2} k \sin 2x = 1$   
 $\Rightarrow \sin 2x = \frac{2}{k}$
- $|\sin 2x| \leq 1 \Rightarrow \left| \frac{2}{k} \right| \leq 1$   
 $\Rightarrow |k| \geq 2$
- b  $3 \cos x - \operatorname{cosec} x = 0 \Rightarrow \sin 2x = \frac{2}{3}$   
 $2x = 41.810, 180 - 41.810, 360 + 41.810, 540 - 41.810$   
 $2x = 41.810, 138.190, 401.810, 498.190$   
 $x = 20.9, 69.1, 200.9, 249.1$  (1dp)

- 6 assume  $x^2 - y^2 = 1$ , where  $x, y \in \mathbb{Z}^+$   
 $x^2 - y^2 = 1 \Rightarrow (x+y)(x-y) = 1$   
 $x, y \in \mathbb{Z}^+ \Rightarrow (x+y), (x-y) \in \mathbb{Z}$  and  $(x+y) > 0$   
 $\therefore x+y = 1$  and  $x-y = 1$   
 adding  $\Rightarrow 2x = 2$   
 $\Rightarrow x = 1$   
 $\Rightarrow y = 0$   
 $\Rightarrow$  contradiction  $\therefore$  no positive integer solutions
- 7 a false e.g.  $a = \sqrt{2}, b = 1 - \sqrt{2} \Rightarrow a$  and  $b$  irrational  
 and  $a + b = 1$  which is rational  
 [ many other examples ]
- b true  $m, n$  consecutive odd integers  $\Rightarrow m = 2a + 1, n = 2a + 3, a \in \mathbb{Z}$   
 $\Rightarrow m + n = 2a + 1 + 2a + 3 = 4a + 4 = 4(a + 1)$   
 $a + 1 \in \mathbb{Z} \therefore m + n$  divisible by 4
- c false e.g.  $x = \frac{5\pi}{3} \Rightarrow \cos x = \frac{1}{2}$  and  $1 + \sin x = 1 - \frac{\sqrt{3}}{2}, \frac{1}{2} > 1 - \frac{\sqrt{3}}{2}$   
 [ any value of  $x$  of the form  $2n\pi + y, n \in \mathbb{Z}, -\frac{\pi}{2} < y < 0$  ]
- 8 a  $\log_2 3 = \frac{p}{q} \Rightarrow 2^{\frac{p}{q}} = 3$   
 $\Rightarrow (2^{\frac{p}{q}})^q = 3^q$   
 $\Rightarrow 2^p = 3^q$
- b assume  $\log_2 3$  is rational  $\Rightarrow \log_2 3 = \frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0$   
 $\Rightarrow 2^p = 3^q$   
 2 and 3 are co-prime  $\Rightarrow p = q = 0$   
 $\Rightarrow$  contradiction  $\therefore \log_2 3$  is irrational
- c e.g.  $a = 2, b = \sqrt{2} \Rightarrow a$  rational and  $b$  irrational  
 and  $\log_a b = \frac{1}{2}$  which is rational  
 [ many other examples ]
- 9 a  $y = \frac{x-2}{4x}$  swap  $x = \frac{y-2}{4y}$   
 $4xy = y - 2$   
 $y(4x - 1) = -2$   
 $y = \frac{2}{1-4x}$   
 $f^{-1}(x) = \frac{2}{1-4x}, x \in \mathbb{R}, x \neq \frac{1}{4}$
- b  $f(x) = f^{-1}(x) \Rightarrow \frac{x-2}{4x} = \frac{2}{1-4x}$   
 $\Rightarrow (x-2)(1-4x) = 8x$   
 $\Rightarrow 4x^2 - x + 2 = 0$   
 $b^2 - 4ac = 1 - 32 = -31$   
 $b^2 - 4ac < 0 \Rightarrow$  no real roots  
 $\therefore$  no real values of  $x$  for which  $f(x) = f^{-1}(x)$